

Hedging in an equilibrium-based model for a large investor. *

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Abstract

We study a financial model with a non-trivial price impact effect. In this model we consider the interaction of a large investor trading in an illiquid security, and a market maker who is quoting prices for this security. We assume that the market maker quotes the prices such that by taking the other side of the investor's demand, the market maker will arrive at maturity with maximal expected wealth. Within this model we concentrate on the issue of contingent claims hedging.

Key words: large investor, liquidity, utility optimization, equilibrium

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1 Introduction

The question of valuation of contingent claims for a small economic agent is well studied in various settings. In the case of complete markets the price of a contingent claim is the initial capital of the replication strategy (a unique arbitrage-free price). For incomplete markets exact replication is rarely possible. In this case the utility-based valuation approach described in the previous section is often used. The basic economic assumption (imposed either implicitly or explicitly) behind the general incomplete model in Mathematical Finance is:

“The agent can trade any security in the *desired* quantity at the *same* price”.

The interpretation of this assumption is that the actions of the agent do not affect prices of securities and that there is no shortage of any security in any quantity.

One way to relax this assumption is to introduce the notion of liquidity into the model. Liquidity is a complex concept standing for *the ease of trading a security*. (Il)liquidity can have different sources, such as inventory risk – Stoll (1978), transaction costs – Cvitanić and Karatzas (1995), uncertain holding horizons – Huang (2003), asymmetry of information – Gârleanu and Pedersen (2004), demand pressure – Gârleanu et al., search friction – Duffie et al. (2005), stochastic supply curve – Çetin et al. (2004) and demand for immediacy – Grossman and Miller (1988), among many others (see Amihud et al. (2005) for a thorough literature overview).

In this paper we will relax the small economic agent assumption by considering a model where agent’s actions move prices. In other words, we shall study a financial model with a non-trivial *price impact* effect. A practical example of such a market is provided, for instance, by an over-the-counter market for an illiquid security, where a market maker quotes prices on demand. In practice it turns out that the price quoted depends on the transaction size. To distinguish our case from the classical one we shall refer to the economic agent trading on such a market as a “large” investor.

We will consider the interaction of a large investor trading in an illiquid security, and a market maker who is quoting prices for this security. We will assume that the market maker quotes the prices such that by taking the other side of the investor’s demand, she will arrive at maturity with maximal expected wealth. This idea was used in a recent paper by Gârleanu et al. for

the discrete time case, but only when the utility function of the market maker is of an exponential form. Using equilibrium-based arguments the authors of that paper considered the question of the evaluation of contingent claims. However, they did not study the question of hedging.

The novelty of our study is that we look at the problem of replication of contingent claims in the model with price impact. Moreover, many of our results are derived in the continuous time framework and with utility functions of rather general form. We will show the existence of a unique pricing rule for a broad class of derivative securities and utility functions, as well as the existence of a unique trading strategy that leads to a perfect replication.

Let us point out that our approach to the model of a large investor follows the traditional framework of Economic Theory. We begin with economic primitives (such as agent's preferences and market equilibrium) and *then* derive the model. This is different from several papers in Mathematical Finance where the nature of illiquidity is postulated *a priori*, see for example, Cvitanić and Ma (1996), Çetin et al. (2004), Bank and Baum (2004) and Frey (1998).

The idea that the price is determined by the zero net supply condition on the market with multiple agents that are solving their individual optimization problems (maximizing their terminal utility) is not new and was studied in the classical paper by Karatzas, Lehoczky and Shreve in Karatzas et al. (1990). Unlike Karatzas et al. (1990) where multiple agents (*small investors/liquid market*) with different utility functions are considered, we consider only one representative agent (*large investor/illiquid market*). This allows us to avoid complicated fixed point arguments used in Karatzas et al. (1990).

This paper is organized as follows. Section 2 defines the basic concepts. Section 3 defines the replication strategy that is suitable for our model and discusses existence and uniqueness of such a strategy. Within this section the assumption of completeness of the market with respect to the price process is playing an important role. Section 4 is devoted to the study of market completeness. This section also contains some particular examples. For instance, here we prove that in the framework of the Bachelier model and under the assumption of an exponential utility for the market maker we can replicate any convex (in appropriate sense) European-type contingent claim (e.g. convex combinations of long calls).

2 Large investor market model

We assume that the uncertainty and the flow of information are modeled by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, where the filtration \mathcal{F} is generated by a J -dimensional Brownian Motion B , that is,

$$(2.1) \quad \mathcal{F}_t = \mathcal{F}_t^B, \quad 0 \leq t \leq T.$$

Here T is a finite time horizon, and $\mathcal{F} = \mathcal{F}_T$.

The security market consists of J risky assets and a riskless asset. These assets are traded between the investor and the market maker. We work in discounted terms and (without loss of generality) assume that the return on the riskless asset is zero. We denote by \mathcal{F}_T -measurable random variables $f = (f^j)_{1 \leq j \leq J}$ the payoffs of the risky assets at maturity and by $S^H = (S_t^H)_{0 \leq t \leq T}$ the (J -dimensional) price process of the risky assets under the condition that the investor is using the (J -dimensional) trading strategy or *demand process* $H = (H_t)_{0 \leq t \leq T}$. Of course, at maturity the price does not depend on the strategy:

$$S_T^H = f, \text{ for all } H.$$

From here on we will implicitly understand that we have J -dimensional processes, and without loss of generality we will use one-dimensional notation. However, in Section 4 we will explicitly point out the vectors and the matrices that appear in the proofs.

The market maker can be viewed then as a *liquidity provider*. She takes the other side of the investor's demand, which can be positive, as well as negative. We assume that the market maker always responds to the investor's demand, that is the market maker always quotes the price (which turns out to be a function of the trade size). The reason for this assumption is that the market maker is naturally forced to quote the prices to achieve the equilibrium by meeting the investor's demand. Of course by equilibrium we mean that both parties are "happy" with the current prices, have no desire to act to change these prices, and the supply is equal to the demand. In order to describe "happiness" of the market maker we use the standard apparatus of utility functions. We assume that the market maker has a utility function $U : \mathcal{R} \rightarrow \mathcal{R}$, which is strictly increasing, strictly concave,

continuously differentiable, and satisfies the Inada conditions

$$\begin{aligned} U'(-\infty) &= \lim_{x \rightarrow -\infty} U'(x) = \infty, \\ U'(\infty) &= \lim_{x \rightarrow \infty} U'(x) = 0. \end{aligned}$$

We shall also require the following two technical assumptions.

Assumption 2.1. *The terminal value of the traded asset $f = (f^j)_{1 \leq j \leq J} \in \mathcal{F}_T$, and the terminal value of the contingent claim $g \in \mathcal{F}_T$ have finite exponential moments, that is*

$$\mathbb{E}[\exp(\langle q, f \rangle)] < \infty, \quad \mathbb{E}[\exp(rg)] < \infty, \quad q \in \mathcal{R}^J, \quad r \in \mathcal{R}.$$

Assumption 2.2. *Utility function $U : \mathcal{R} \rightarrow \mathcal{R}$ satisfies*

$$(2.2) \quad c_1 < -\frac{U'(x)}{U''(x)} < c_2 \text{ for some } c_1, c_2 > 0.$$

Clearly, a linear combination of exponential functions of the form

$$U(x) = \sum_{i=1}^N -c_i \frac{e^{-\gamma_i x}}{\gamma_i}, \quad \gamma_i, c_i > 0, \quad x \in \mathcal{R}$$

satisfies the assumption above.

Notice that Assumption 2.2 implies the Inada conditions.

We assume that the investor reveals his market orders (his demand process) H to the market maker. The market maker responds to the investor's demand by quoting the price, and by taking the other side of the demand. That is, if H is the investor's strategy, then $-H$ is the market maker's strategy. In other words, the market maker responds to the demand so that the market rests in equilibrium (supply equals demand). The market maker is quoting the price in such a way that she arrives at maturity with maximal expected wealth. Formally this can be stated as

Definition 2.1. *Let $x \in \mathcal{R}$ be the initial cash endowment of the market maker. Let $f = (f^j)_{1 \leq j \leq J}$ be an \mathcal{F}_T -measurable contingent claim. Let $H = (H^j)_{1 \leq j \leq J}$ be a predictable process. The equivalent probability measure $\mathbb{P}^H \sim \mathbb{P}$ is called the pricing measure of f under demand H , and the semimartingale S^H is called the price process of f under demand H if*

$$(2.3) \quad \frac{d\mathbb{P}^H}{d\mathbb{P}} \triangleq \frac{U'(x - \int_0^T H_u dS_u^H)}{\mathbb{E}[U'(x - \int_0^T H_u dS_u^H)]},$$

and the price process S^H with the integral $\int H dS^H$ are martingales under \mathbb{P}^H . In particular,

$$S_t^H \triangleq \mathbb{E}^H[f|\mathcal{F}_t], \quad 0 \leq t \leq T.$$

Notice that the Definition 2.1 is rather general, as it does not specify any conditions on the utility function U , the demand process H , and the contingent claim f .

The above definition displays an intimate relationship between the price process and the pricing measure. It may not be clear from the formulation of Definition 2.1 that it reflects the mechanics of the market described in the previous paragraph. However, notice that the density of \mathbb{P}^H is chosen in such a way that the process $-H$ is indeed a solution to the market maker's optimization problem (which will be defined below.) Naturally, the semi-martingale S^H is defined in such a way that it is a martingale under the pricing measure. It will become evident from the following lemma, that the numerator of (2.3) is nothing else but the market maker's marginal utility.

Lemma 2.1. *Let $x \in \mathcal{R}$ be the initial cash endowment of the market maker. Suppose f satisfies Assumption 2.1, and U satisfies Assumption 2.2. Let $H = (H^j)_{1 \leq j \leq J}$ be a predictable process. Suppose that S^H is the price process of f under demand H . Then $-H$ is the unique solution of the optimization problem*

$$(2.4) \quad u(x) \triangleq \max_{G \in \mathcal{H}(S^H, \mathbb{P}^H)} \mathbb{E}[U(x + \int_0^T G_u dS_u^H)],$$

where $\mathcal{H}(S^H, \mathbb{P}^H)$ is the collection of predictable processes G such that

$$\int_0^T G_u dS_u^H$$

is a \mathbb{P}^H -martingale.

Proof. Let

$$\mathcal{A}(x) \triangleq \{h : \mathbb{E}^H[h] \leq x\}.$$

In order to prove that $-H$ is the unique solution of the optimization problem (2.4), we need to show that

$$\hat{h} \triangleq x - \int_0^T H_u dS_u^H$$

is an element of $\mathcal{A}(x)$, and for any $h \in \mathcal{A}(x)$, the following inequality holds true

$$(2.5) \quad \mathbb{E}[U(h)] \leq \mathbb{E}[U(\hat{h})].$$

The fact that \hat{h} is an element of $\mathcal{A}(x)$ follows from the martingale property of $\int H dS^H$ under \mathbb{P}^H , see Definition 2.1.

Further, let $V(y)$ be the Legendre transform of $U(z)$, i.e.

$$(2.6) \quad V(y) \triangleq \sup_{z \in \mathcal{R}} \{U(z) - zy\}, \quad y > 0$$

It follows from (2.6) that for any $y \geq 0$ and $z \in \mathcal{R}$

$$U(z) \leq V(y) + zy,$$

and therefore

$$(2.7) \quad \begin{aligned} \mathbb{E}[U(h)] &\leq \mathbb{E} \left[V \left(y \frac{d\mathbb{P}^H}{d\mathbb{P}} \right) \right] + \mathbb{E} \left[hy \frac{d\mathbb{P}^H}{d\mathbb{P}} \right] = \mathbb{E} \left[V \left(y \frac{d\mathbb{P}^H}{d\mathbb{P}} \right) \right] + \mathbb{E}^H[h]y \\ &\leq \mathbb{E} \left[V \left(y \frac{d\mathbb{P}^H}{d\mathbb{P}} \right) \right] + xy, \end{aligned}$$

where the last inequality follows by virtue of h being an element of $\mathcal{A}(x)$. On the other hand, the identity

$$U(I(y)) = V(y) + yI(y), \quad \text{where } I(y) = (U')^{-1}(y), \quad y > 0$$

along with

$$I \left(y \frac{d\mathbb{P}^H}{d\mathbb{P}} \right) = (U')^{-1} \left(y \frac{d\mathbb{P}^H}{d\mathbb{P}} \right) = \hat{h}, \quad y = \mathbb{E}[U'(\hat{h})]$$

implies that

$$(2.8) \quad \begin{aligned} \mathbb{E}[U(\hat{h})] &= \mathbb{E} \left[V \left(y \frac{d\mathbb{P}^H}{d\mathbb{P}} \right) \right] + \mathbb{E} \left[\hat{h} y \frac{d\mathbb{P}^H}{d\mathbb{P}} \right] = \mathbb{E} \left[V \left(y \frac{d\mathbb{P}^H}{d\mathbb{P}} \right) \right] + \mathbb{E}^H[\hat{h}]y \\ &= \mathbb{E} \left[V \left(y \frac{d\mathbb{P}^H}{d\mathbb{P}} \right) \right] + xy. \end{aligned}$$

Now we compare (2.7) with (2.8) and conclude that (2.5) holds true, and therefore

$$u(x) = \mathbb{E}[U(\hat{h})] = \mathbb{E}[U(x - \int_0^T H_u dS_u^H)].$$

□

In what follows we are interested to find the answers to the following question: Is it possible for a large trader to replicate another *non-traded* contingent claim g , that is, form a demand H such that for some initial wealth p

$$p + \int_0^T H_u dS_u^H = g?$$

There are another two important questions to ask:

- Does the price process S^H exist for an arbitrary demand H ?
- Provided that S^H exists, is it unique?

The answers to the two latter questions are given in the companion paper by German (Forthcoming), while in this paper we will be concerned with the former question.

3 Replication

Consider an \mathcal{F}_T -measurable random variable g (alternatively we will call it a *non-traded* European-type contingent claim). The utmost important question is what is the “fair” price of this claim. We remind the reader, that in the framework of complete financial model for a small economic agent the arbitrage-free price of g is given by

$$p = \mathbb{E}_{\mathbb{P}^*}[g]$$

and the unique hedging strategy can be found from the martingale representation

$$g = p + \int_0^T H_t dS_t,$$

when S is a martingale under the unique martingale measure \mathbb{P}^* .

The classical theory of asset pricing hinges on the crucial assumption that the price per share of an asset does not depend on the size of the trade at any moment in time. Moreover, when pricing by replication, it is understood that the integrand and the integrator of the wealth process are not functions of each other. In our large trader model the price process *is* a (non-linear) function of demand. Therefore the problem of replication in the illiquid market cannot be solved using the tools of the classical asset pricing theory.

In order to construct a perfect hedge for a non-traded contingent claim it has to be taken into account that there is a back-and-forth relationship between the size of the trade and the current price of the traded asset. More precisely, we have the following definition.

Definition 3.1. *Let $g, f = (f^j)_{1 \leq j \leq J}$ be \mathcal{F}_T -measurable random variables. A predictable process H is called a hedging strategy of g , if there exist $p \in \mathcal{R}$, and a price process S^H of f under demand H such that*

$$g = p + \int_0^T H_u dS_u^H.$$

Remark 3.1. Similarly to pricing by replication in the classical framework of a small economic agent, we will call p a *price of g* , since p is the initial capital required for the perfect replication of g . Note that it is not clear *a priori* that p is defined uniquely. We shall show below that the uniqueness always holds true for exponential utilities.

Remark 3.2. The above definition looks similar to the classical definition of a hedging strategy, with the crucial difference that the price process and the hedging strategy depend on each other.

Theorem 3.1 (Necessary condition). *Let $x \in \mathcal{R}$ be the initial capital of the market maker. Assume that f and g satisfy Assumption 2.1, and U satisfies Assumption 2.2. Suppose there exists a hedging strategy H of the contingent claim g with price p . Then the unique pricing measure \mathbb{P}^H is given by the density*

$$(3.1) \quad \frac{d\mathbb{P}^H}{d\mathbb{P}} = \frac{U'(x + p - g)}{\mathbb{E}[U'(x + p - g)]},$$

the price process is unique and is given by

$$S_t^H = \mathbb{E}^H[f|\mathcal{F}_t].$$

Moreover, the following integral representation holds true

$$(3.2) \quad \mathbb{E}^H[g|\mathcal{F}_t] = p + \int_0^t H_u dS_u^H, \text{ for any } t \in [0, T].$$

Remark 3.3. From integral representation (3.2) we deduce that the hedging process H is defined uniquely a.s. on $\Omega \times [0, T]$ with respect to $d\mathbb{P}[\omega] \times d\langle S^H \rangle_t$ in the following sense. Due to the Assumption 2.1, g has finite exponential moments and therefore it is also square-integrable. Therefore if there exists another hedging strategy \tilde{H} such that

$$\mathbb{E}^H[g|\mathcal{F}_t] = p + \int_0^t \tilde{H}_u dS_u^{\tilde{H}},$$

then

$$\sum_{j=1}^J \int_0^T |\tilde{H}_t^j - H_t^j|^2 d\langle S^{H^j} \rangle_t = 0 \quad \text{a.s.}$$

Proof. Since H is a hedging strategy of the contingent claim g , there exists S^H , the price process of f under demand H along with the corresponding pricing measure \mathbb{P}^H , such that

$$(3.3) \quad g = p + \int_0^T H_u dS_u^H.$$

Therefore by Definition 2.1,

$$(3.4) \quad \frac{d\mathbb{P}^H}{d\mathbb{P}} = \frac{U'(x - \int_0^T H_u dS_u^H)}{\mathbb{E}[U'(x - \int_0^T H_u dS_u^H)]} = \frac{U'(x + p - g)}{\mathbb{E}[U'(x + p - g)]}.$$

Due to Assumption 2.2, the first derivative of the utility function U is bounded from below and above by exponential functions. Therefore Assumption 2.1 implies that the random variable (3.4) (the density of the pricing measure \mathbb{P}^H) is well defined. It is also unique, and so is the price process S^H , which by Definition 2.1 is equal to

$$S_t^H = \mathbb{E}^H[f|\mathcal{F}_t], \quad 0 \leq t \leq T.$$

Since $\int_0^t H_u dS_u^H$ is a \mathbb{P}^H -martingale, by applying conditional expectation to the both sides of (3.3) we obtain

$$\mathbb{E}^H[g|\mathcal{F}_t] = p + \int_0^t H_u dS_u^H, \quad \text{for any } t \in [0, T],$$

along with

$$\mathbb{E}^H[g] = p.$$

□

We start the study of the existence of replication strategy with the following

Lemma 3.1. *Assume that g satisfies Assumption 2.1, U satisfies Assumption 2.2, and (2.1) holds true. Then for any $x \in \mathcal{R}$ the equation*

$$(3.5) \quad \mathbb{E}[(p - g)U'(x + p - g)] = 0$$

has a solution. If, in addition, the utility function U is exponential, that is,

$$U(x) = -\frac{1}{\gamma}e^{-\gamma x}$$

for some $\gamma > 0$, then the solution of (3.5) is unique and given by

$$p = \frac{\mathbb{E}[ge^{-\gamma g}]}{\mathbb{E}[e^{-\gamma g}]}.$$

Proof. Let us consider the function $\alpha(p) : \mathcal{R} \rightarrow \mathcal{R}$ defined as

$$\alpha(p) \triangleq \mathbb{E} \left[(p - g) \frac{U'(x + p - g)}{U'(x + p)} \right].$$

We will show that $\alpha(p)$ has a zero (at least one). Since U is a utility function (strictly increasing, strictly concave), $U'(x + p)$ is non-random and strictly positive for all x and p . Therefore the existence of a solution of the equation

$$(3.6) \quad \alpha(p) = 0$$

will imply the existence of a solution of (3.5).

In order to show the existence of a solution of (3.6) it is sufficient to show that

$$(3.7) \quad \lim_{p \rightarrow -\infty} \alpha(p) \leq 0, \text{ and}$$

$$(3.8) \quad \lim_{p \rightarrow \infty} \alpha(p) \geq 0.$$

It follows directly from Assumption 2.2 that for any $z, y \in \mathcal{R}$

$$e^{-c_2 y} \leq \frac{U'(z + y)}{U'(z)} \leq e^{-c_1 y}.$$

Therefore

$$e^{c_2 g} \leq \frac{U'(x+p-g)}{U'(x+p)} \leq e^{c_1 g},$$

and due to Assumption 2.1, the random variable $\frac{U'(x+p-g)}{U'(x+p)}$ has a finite positive expectation and is square-integrable. Hence by Hölder's inequality

$$\mathbb{E} \left[g \frac{U'(x+p-g)}{U'(x+p)} \right] < \infty,$$

and therefore

$$(3.9) \quad \begin{aligned} \lim_{p \rightarrow -\infty} \alpha(p) &= \lim_{p \rightarrow -\infty} \mathbb{E} \left[(p-g) \frac{U'(x+p-g)}{U'(x+p)} \right] = -\infty, \\ \lim_{p \rightarrow \infty} \mathbb{E} \left[p \frac{U'(x+p-g)}{U'(x+p)} \right] &= \infty, \end{aligned}$$

$$(3.10) \quad \lim_{p \rightarrow \infty} \mathbb{E} \left[g \frac{U'(x+p-g)}{U'(x+p)} \right] = C < \infty,$$

for some constant $C \in \mathcal{R}$. As a direct consequence of (3.9) and (3.10) we obtain

$$\lim_{p \rightarrow \infty} \alpha(p) = \lim_{p \rightarrow \infty} \mathbb{E} \left[(p-g) \frac{U'(x+p-g)}{U'(x+p)} \right] = \infty.$$

Therefore we conclude that (3.6) has a solution and consequently (3.5) has a solution.

Equation (3.5) can be written as

$$p = \frac{\mathbb{E}[gU'(x+p-g)]}{\mathbb{E}[U'(x+p-g)]},$$

and for an exponential utility function we have

$$p = \frac{\mathbb{E}[g \exp\{-\gamma(x+p-g)\}]}{\mathbb{E}[\exp\{-\gamma(x+p-g)\}]} = \frac{\mathbb{E}[ge^{\gamma g}]}{\mathbb{E}[e^{\gamma g}]},$$

which is the unique solution of (3.5). \square

Remark 3.4. In the following theorem we will need the notion of *completeness*. By completeness we will understand that every European-type derivative security (an \mathcal{F}_T -measurable random variable) can be represented as a constant plus an integral with respect to the underlying security.

We shall now state

Theorem 3.2 (Sufficient condition). *Let $x \in \mathcal{R}$ be the initial capital of the market maker. Assume that f and g satisfy Assumption 2.1, U satisfies Assumption 2.2, and (2.1) holds true. Let \mathbb{P} be a probability measure such that for some $p \in \mathcal{R}$*

$$(3.11) \quad \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \triangleq \frac{U'(x + p - g)}{\mathbb{E}[U'(x + p - g)]},$$

and

$$(3.12) \quad \tilde{\mathbb{E}}[g] = p.$$

Let us denote

$$(3.13) \quad \tilde{S}_t = \tilde{\mathbb{E}}[f | \mathcal{F}_t],$$

and suppose that the model is complete with respect to \tilde{S} . Then there exists a hedging strategy H such that

$$g = p + \int_0^T H_u dS_u^H,$$

with $S^H = \tilde{S}$, the price process of f under demand H .

Remark 3.5. The constant p allowing for the existence of pricing measure $\tilde{\mathbb{P}}$ satisfying (3.11) and (3.12) is exactly the one solving (3.5). Note that, by Lemma 3.1, p is defined uniquely for an important class of exponential utilities.

Proof. First we observe that since the market is complete with respect to \tilde{S} , the probability measure $\tilde{\mathbb{P}}$ defined by (3.11) is the unique martingale measure of \tilde{S} .

Since the market is complete with respect to \tilde{S} , any \mathcal{F}_T -measurable random variable can be represented as a constant plus an integral with respect to \tilde{S} . Moreover, due to Assumption 2.1, random variable g is square-integrable, and therefore there exists a predictable process H , such that the process $\int_0^t H_u d\tilde{S}_u$ is a $\tilde{\mathbb{P}}$ -martingale, and

$$(3.14) \quad g = \tilde{\mathbb{E}}[g] + \int_0^T H_u d\tilde{S}_u = p + \int_0^T H_u d\tilde{S}_u,$$

where the last equality follows from (3.12). Hence

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{U'(x + p - g)}{\mathbb{E}[U'(x + p - g)]} = \frac{U'(x - \int_0^T H_u d\tilde{S}_u)}{\mathbb{E}[U'(x - \int_0^T H_u d\tilde{S}_u)]}.$$

By Definition 2.1, the above implies that $\mathbb{P}^H = \tilde{\mathbb{P}}$ is the unique pricing measure of f under demand H , and $S^H = \tilde{S}$ is the unique price process of f under demand H . Hence, by Definition 3.1 H is the hedging strategy of g . \square

Remark 3.6. We point out again that the integral representation (3.14) implies that the hedging process H is defined uniquely a.s. on $\Omega \times [0, T]$ with respect to $d\mathbb{P}[\omega] \times d\langle S^H \rangle_t$ in the sense explained in Remark 3.3.

The assumption on market completeness is essential for the sufficient condition. Here we present an example of an incomplete market for which the above theorem does not hold, and there is no hedging strategy for a particular contingent claim g .

Example 3.1. Consider the case of a one-dimensional Brownian Motion B . Let $g = B_T$, and let $U(x) = -e^{-x}$. Therefore

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{e^{-B_T}}{e^{\frac{1}{2}T}} = e^{-B_T - \frac{1}{2}T},$$

and by Girsanov Theorem under the probability measure $\tilde{\mathbb{P}}$ there exists another Brownian Motion \tilde{B} such that

$$\tilde{B}_t = B_t + t.$$

Further, let

$$f = B_T - B_\tau + T - \tau = \int_0^T \mathbf{1}_{[\tau, T]}(u) d\tilde{B}_u$$

for some discrete time $0 < \tau < T$. In this case

$$\begin{aligned} \tilde{S}_t &= \mathbb{E}\left[\int_0^T \mathbf{1}_{[\tau, T]}(u) d\tilde{B}_u \middle| \mathcal{F}_t\right] \\ &= \int_0^t \mathbf{1}_{[\tau, T]}(u) d\tilde{B}_u = \int_0^t \sigma_u d\tilde{B}_u, \quad \sigma_t = \mathbf{1}_{[\tau, T]}(t), \end{aligned}$$

which implies that the process \tilde{S} is identically zero on the time interval $[0, \tau)$. There exists a martingale representation of

$$g = -B_T = -T + \int_0^T d\tilde{B}_u$$

as an integral with respect to the Brownian Motion \tilde{B} . However, it is impossible to represent g as an integral with respect to \tilde{S} , since the volatility σ is zero on $[0, \tau]$. In other words, the model is incomplete with respect to \tilde{S} .

□

4 Completeness with respect to \tilde{S} .

As it was highlighted by Example 3.1, for the existence of a replication strategy it is necessary to verify that the market driven by the price process \tilde{S} is complete. This problem will be the focus of the current section.

We start by recalling (without the proof) the following (well-known) fact lying in the intersection of the Girsanov and the Martingale Representation theorems.

Lemma 4.1. *Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ is generated by a J -dimensional Brownian motion B . Let $\tilde{\mathbb{P}} \sim \mathbb{P}$ and let $\alpha = (\alpha_t)$ be a J -dimensional stochastic process such that*

$$(4.1) \quad \tilde{B}_t \triangleq B_t + \int_0^t \alpha_u du, \quad 0 \leq t \leq T,$$

is the J -dimensional Brownian Motion under $\tilde{\mathbb{P}}$. Then any $\tilde{\mathbb{P}}$ -martingale \tilde{M} is a stochastic integral with respect to \tilde{B} .

Proposition 4.1. *Assume that f and g satisfy Assumption 2.1, U satisfies Assumption 2.2 and that the filtration is generated by the Brownian motion B (that is, (2.1) holds true). Let $\tilde{\mathbb{P}}$ be the probability measure defined in (3.11), \tilde{S} be the price process defined in (3.13) and \tilde{B} be the J -dimensional Brownian motion under $\tilde{\mathbb{P}}$ given by (4.1).*

Then the financial model determined by the price process \tilde{S} given by (3.13) is complete if and only if the $J \times J$ -dimensional matrix process $\tilde{\sigma} = (\tilde{\sigma}_t)$ in the martingale representation

$$(4.2) \quad \tilde{S}_t = \tilde{S}_0 + \int_0^t \tilde{\sigma}_u \cdot d\tilde{B}_u$$

or, component-wise,

$$\tilde{S}_t^i = \tilde{S}_0^i + \sum_{1 \leq j \leq J} \int_0^t \tilde{\sigma}_u^{ij} d\tilde{B}_u^j, \quad 1 \leq i \leq J,$$

has full rank almost everywhere with respect to the product measure $d\mathbb{P}[\omega] \times dt$.

Proof. Let us assume that the matrix $\tilde{\sigma}$ has full rank (or in other words it is invertible) almost everywhere with respect to the product measure $d\mathbb{P}[\omega] \times dt$. Let Γ be an \mathcal{F}_T -measurable J -dimensional random variable. Then by Lemma 4.1 there exists an adapted $J \times J$ -dimensional matrix process $\gamma = (\gamma_t)_{0 \leq t \leq T}$ such that

$$\Gamma = \tilde{\mathbb{E}}[\Gamma] + \int_0^T \gamma_u \cdot d\tilde{B}_u.$$

Since $\tilde{\sigma}$ is invertible $d\mathbb{P}[\omega] \times dt$ -almost everywhere, and by (4.2), the above expression is equal to

$$\Gamma = \tilde{\mathbb{E}}[\Gamma] + \int_0^T \gamma_u \cdot \tilde{\sigma}_u^{-1} \cdot d\tilde{S}_u.$$

Hence we conclude that the market is complete with respect to \tilde{S} , since every contingent claim is replicable.

The proof of the converse statement will be done by contradiction. Assume that the market is complete with respect to \tilde{S} , and suppose that the matrix $\tilde{\sigma}_t$ is not invertible $d\mathbb{P}[\omega] \times dt$ -almost everywhere. Let us define the following adapted vector process $a = (a_t^j)_{0 \leq t \leq T}$, $1 \leq j \leq J$. For each $0 \leq t \leq T$, let the vector a_t be equal to a non-trivial vector from the null-space of the matrix $\tilde{\sigma}_t$. Since we assumed that $\tilde{\sigma}_t$ is not invertible, the null-space of $\tilde{\sigma}_t$ is not trivial. That is

$$a_t = n_t, \quad n_t \in \text{Null}(\tilde{\sigma}_t), \quad n_t \neq \vec{0},$$

where $\text{Null}(\tilde{\sigma}_t)$ is the Null-space of $\tilde{\sigma}_t$, and $\vec{0}$ is a J -dimensional zero vector. Let us consider two J -dimensional $\tilde{\mathbb{P}}$ -martingales

$$\int_0^t a_u d\tilde{B}_u, \text{ and } \int_0^t \tilde{\sigma}_u \cdot d\tilde{B}_u,$$

or component-wise,

$$\int_0^t a_u^k d\tilde{B}_u^k, \text{ and } \sum_{1 \leq j \leq J} \int_0^t \tilde{\sigma}_u^{kj} d\tilde{B}_u^j, \quad 1 \leq k \leq J.$$

Due to the choice of a , at any time t the dot-product of a_t and $\tilde{\sigma}_t^T$ is a zero vector, and therefore their cross-variation is

$$\langle \int_0^\cdot a_u d\tilde{B}_u, \int_0^\cdot \tilde{\sigma}_u \cdot d\tilde{B}_u \rangle_t = \int_0^t a_u \cdot \tilde{\sigma}_u^T du = \vec{0}, \quad 0 \leq t \leq T,$$

as well as

$$\langle \int_0^\cdot a_u d\tilde{B}_u, \tilde{S} \cdot \rangle_t = \vec{0}.$$

Since the cross-variation process of two $\tilde{\mathbb{P}}$ -martingales is zero, the product of these martingales is a zero process, which implies that $\int_0^t a_u d\tilde{B}_u$ and \tilde{S}_t are orthogonal. Notice that the random variable

$$(4.3) \quad \int_0^T a_u d\tilde{B}_u$$

is different from zero because by the assumption the matrix $\tilde{\sigma}$ does not have full rank almost everywhere with respect to the product measure $d\mathbb{P}[\omega] \times dt$. We assumed that the market is complete and therefore each non-trivial \mathcal{F}_T -measurable random variable is non-trivially replicable. However, due to orthogonality of $\int_0^t a_u d\tilde{B}_u$ and \tilde{S}_t , the \mathcal{F}_T -measurable random variable (4.3) is not \tilde{S} -replicable, which leads us to a contradiction. Therefore the market is incomplete with respect to \tilde{S} . \square

The computation of the $J \times J$ -matrix volatility process $\tilde{\sigma} = (\tilde{\sigma}_t)$ can often be done with the help of the Clark-Ocone formula and the Malliavin calculus as is illustrated in the following lemma. For a random variable ψ we denote by $\mathbf{D}_t(\psi) = (\mathbf{D}_t^j(\psi))_{1 \leq j \leq J}$ the Malliavin derivative of ψ at time t with respect to the Brownian Motion B .

Lemma 4.2. *In addition to conditions of Proposition 4.1 assume that f and g satisfy Assumption 2.1 and are Malliavin differentiable.*

Then the matrix $\tilde{\sigma}_t$ in the integral representation (4.2) is given by

$$(4.4) \quad \tilde{\sigma}_t^{ij} = \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i) + A(x + p - g)(f^i - \tilde{S}_t^i)\mathbf{D}_t^j(g)|\mathcal{F}_t], \quad 1 \leq i, j \leq J,$$

where $A = A(x)$ is the absolute risk-aversion coefficient of U given by

$$(4.5) \quad A(x) \triangleq -\frac{U''(x)}{U'(x)}, \quad x \in \mathcal{R}.$$

Proof. Our goal is to compute the integrand in the form of a $J \times J$ -dimensional matrix process in the martingale representation (4.2) of the process \tilde{S} .

Since the density process is $Z_t = \mathbb{E}\left[\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \middle| \mathcal{F}_t\right]$, it can be also expressed as

$$(4.6) \quad Z_t = \mathbb{E}\left[\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \middle| \mathcal{F}_t\right] = \frac{\mathbb{E}[U'(x + p - g)|\mathcal{F}_t]}{\mathbb{E}[U'(x + p - g)]}.$$

Let us introduce another J -dimensional \mathbb{P} -martingale. Let

$$R_t = \mathbb{E}[Z_T \tilde{S}_T | \mathcal{F}_t].$$

The density process is an exponential martingale and so for some adapted J -dimensional process $\alpha = (\alpha_t)_{0 \leq t \leq T}$ we have $Z_t = \mathcal{E}(\alpha_t \cdot B_t)$, where \mathcal{E} is the stochastic exponent. Since

$$\tilde{S}_t = \tilde{\mathbb{E}}[f | \mathcal{F}_t] = \frac{1}{Z_t} \mathbb{E}[Z_T f | \mathcal{F}_t],$$

it follows that $R_t = Z_t \tilde{S}_t$. The process R is a \mathbb{P} -martingale, and since the density process Z is strictly positive \mathbb{P} -almost surely, it has the martingale representation

$$(4.7) \quad R_t = \tilde{S}_0 + \int_0^t Z_u \Sigma_u \cdot dB_u$$

for some progressively measurable matrix process Σ . By differentiating $Z_t \tilde{S}_t$ we obtain that the differential of R is

$$(4.8) \quad dR_t = Z_t \tilde{\sigma}_t \cdot dB_t + Z_t \tilde{S}_t \cdot \alpha_t \cdot dB_t.$$

By comparing (4.8) and the differential of (4.7) we obtain

$$(4.9) \quad \tilde{\sigma}_t = \Sigma_t - \tilde{S}_t \cdot \alpha_t.$$

Notice that the above expression for $\tilde{\sigma}$ has two processes Σ and α that are only known to exist and to be unique (in the sense discussed in Remark 3.3), but are not known explicitly.

Due to the assumption that the random variable g has exponential moments and is Malliavin differentiable and using the fact that $dZ_t = Z_t \alpha_t \cdot dB_t$ we deduce from Clark-Ocone formula that

$$(4.10) \quad Z_t \alpha_t^j = \mathbb{E}[\mathbf{D}_t^j(Z_T) | \mathcal{F}_t], \quad 1 \leq j \leq J.$$

Similarly (4.7) implies that

$$Z_t \Sigma_t^{ij} = \mathbb{E}[\mathbf{D}_t^j(Z_T f^i) | \mathcal{F}_t] = \mathbb{E}[Z_T \mathbf{D}_t^j(f^i) + f^i \mathbf{D}_t^j(Z_T) | \mathcal{F}_t],$$

and therefore

$$(4.11) \quad \Sigma_t^{ij} = \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i) | \mathcal{F}_t] + \mathbb{E} \left[\frac{1}{Z_t} f^i \mathbf{D}_t^j(Z_T) \middle| \mathcal{F}_t \right].$$

Hence by combining (4.9), (4.10), (4.11) we obtain

$$(4.12) \quad \begin{aligned} \tilde{\sigma}_t^{ij} &= \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i) | \mathcal{F}_t] + \mathbb{E} \left[\frac{1}{Z_t} f^i \mathbf{D}_t^j(Z_T) \middle| \mathcal{F}_t \right] - \tilde{S}_t^i \mathbb{E} \left[\frac{1}{Z_t} \mathbf{D}_t^j(Z_T) \middle| \mathcal{F}_t \right] \\ &= \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i) | \mathcal{F}_t] + \mathbb{E} \left[\frac{1}{Z_t} (f^i - \tilde{S}_t^i) \mathbf{D}_t^j(Z_T) \middle| \mathcal{F}_t \right]. \end{aligned}$$

Now let us use the assumption that the random variable g is Malliavin differentiable. Then from 4.6 it follows that

$$\mathbf{D}_t^j Z_T = \frac{-U''(x+p-g) \mathbf{D}_t^j(g)}{\mathbb{E}[U'(x+p-g)]} = \frac{A(x+p-g) U'(x+p-g) \mathbf{D}_t^j(g)}{\mathbb{E}[U'(x+p-g)]},$$

where $A(x)$ is the absolute risk-aversion coefficient as defined in (4.5). Therefore the last term of (4.12) can be expressed as

$$\mathbb{E} \left[\frac{1}{Z_t} (f^i - \tilde{S}_t^i) \mathbf{D}_t^j(Z_T) \middle| \mathcal{F}_t \right] = \tilde{\mathbb{E}}[A(x+p-g)(f^i - \tilde{S}_t^i) \mathbf{D}_t^j(g) | \mathcal{F}_t].$$

We can now put everything together to obtain

$$\tilde{\sigma}_t^{ij} = \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i) + A(x+p-g)(f^i - \tilde{S}_t^i) \mathbf{D}_t^j(g) | \mathcal{F}_t].$$

□

The expression (4.4) for $\tilde{\sigma}$ can be simplified if we assume that U is the exponential utility function, and that the contingent claim g is a *standard* European option on f . In the formulation of the result we shall use the notation

$$(4.13) \quad \widetilde{\text{Cov}}(\alpha, \beta | \mathcal{F}_t) = \widetilde{\mathbb{E}}[\alpha\beta | \mathcal{F}_t] - \widetilde{\mathbb{E}}[\alpha | \mathcal{F}_t]\widetilde{\mathbb{E}}[\beta | \mathcal{F}_t]$$

for the conditional covariance of the random variables α and β with respect to the pricing measure $\tilde{\mathbb{P}}$ and the information at time t .

Lemma 4.3. *In addition to conditions of Lemma 4.2 assume that*

$$g = G(f)$$

for some almost everywhere differentiable function $G : \mathcal{R}^J \longrightarrow \mathcal{R}$, and that the utility function $U = U(x)$ is of exponential form:

$$U(x) = \frac{1}{\gamma} e^{-\gamma x}, \quad x \in \mathcal{R},$$

for some $\gamma > 0$. Then the matrix $\tilde{\sigma}_t$ in the integral representation (4.2) is given by

$$(4.14) \quad \tilde{\sigma}_t^{ij} = \widetilde{\mathbb{E}}[\mathbf{D}_t^j(f^i) + \gamma(f^i - \tilde{S}_t^i) \sum_{1 \leq k \leq J} \mathbf{D}_t^j(f^k) \frac{\partial}{\partial x^k} G(f) | \mathcal{F}_t]$$

$$(4.15) \quad = \widetilde{\mathbb{E}}[\mathbf{D}_t^j(f^i) | \mathcal{F}_t] + \gamma \widetilde{\text{Cov}}(f^i, \sum_{1 \leq k \leq J} \mathbf{D}_t^j(f^k) \frac{\partial}{\partial x^k} G(f) | \mathcal{F}_t),$$

$$1 \leq i, j \leq J.$$

Proof. First we remind that for the exponential utility $U(x) = \frac{1}{\gamma} e^{-\gamma x}$, the absolute risk-aversion coefficient $A(x)$ is constant and is equal to

$$A(x) = \gamma, \text{ for every } x \in \mathcal{R}.$$

Equation (4.14) follows directly from (4.4) and from the assumption that g is an almost everywhere differentiable function of f ,

$$\mathbf{D}_t^j(g) = \mathbf{D}_t^j(G(f)) = \sum_{1 \leq k \leq J} \mathbf{D}_t^j(f^k) \frac{\partial}{\partial x^k} G(f), \quad 1 \leq j \leq J,$$

which leads to

$$\begin{aligned}\tilde{\sigma}_t^{ij} &= \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i) + A(x + p - g)(f^i - \tilde{S}_t^i)\mathbf{D}_t^j(g)|\mathcal{F}_t] \\ &= \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i) + \gamma(f^i - \tilde{S}_t^i) \sum_{1 \leq k \leq J} \mathbf{D}_t^j(f^k) \frac{\partial}{\partial x^k} G(f)|\mathcal{F}_t].\end{aligned}$$

As for (4.15), we have

$$\begin{aligned}\tilde{\sigma}_t^{ij} &= \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i) + A(x + p - g)(f^i - \tilde{S}_t^i)\mathbf{D}_t^j(g)|\mathcal{F}_t] = \\ &= \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i)|\mathcal{F}_t] + \gamma\tilde{\mathbb{E}}[f^i\mathbf{D}_t^j(g)|\mathcal{F}_t] - \gamma\tilde{\mathbb{E}}[\tilde{S}_t^i\mathbf{D}_t^j(g)|\mathcal{F}_t] \\ &= \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i)|\mathcal{F}_t] + \gamma\tilde{\mathbb{E}}[f^i\mathbf{D}_t^j(g)|\mathcal{F}_t] - \gamma\tilde{S}_t^i\tilde{\mathbb{E}}[\mathbf{D}_t^j(g)|\mathcal{F}_t] \\ &= \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i)|\mathcal{F}_t] + \gamma\tilde{\mathbb{E}}[f^i\mathbf{D}_t^j(g)|\mathcal{F}_t] - \gamma\tilde{\mathbb{E}}[f^i|\mathcal{F}_t]\tilde{\mathbb{E}}[\mathbf{D}_t^j(g)|\mathcal{F}_t] \\ &= \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i)|\mathcal{F}_t] + \gamma\widetilde{\text{Cov}}(f^i, \mathbf{D}_t^j(g)|\mathcal{F}_t) \\ &= \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i)|\mathcal{F}_t] + \gamma\widetilde{\text{Cov}}(f^i, \sum_{1 \leq k \leq J} \mathbf{D}_t^j(f^k) \frac{\partial}{\partial x^k} G(f)|\mathcal{F}_t).\end{aligned}$$

□

The computation of the matrix $\tilde{\sigma}$ becomes particularly simple in the case when the payoffs of traded contingents claims f are the terminal values of the Brownian Motions B , that is,

$$(4.16) \quad f^j = B_T^j, \quad 1 \leq j \leq J.$$

Note that in this case, in the absence of any trading by the large investor, that is, in the case $H = 0$ the price process of the stocks is given by B . In other words, in the absence of the large investor we have the multi-dimensional Bachelier model. In the statement of the next lemma we use the standard notation

$$\delta_{ij} = 1_{\{i=j\}}$$

for the Kronecker delta.

Lemma 4.4. *In addition to conditions of Lemma 4.3 assume (4.16). Then*

$$(4.17) \quad \tilde{\sigma}_t^{ij} = \delta_{ij} + \gamma\widetilde{\text{Cov}}\left(B_T^i, \frac{\partial}{\partial x^j} G(B_T) \middle| \mathcal{F}_t\right), \quad 1 \leq i, j \leq J.$$

Proof. By the assumption that the payoffs of the contingent claims are Brownian Motions,

$$\tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i)|\mathcal{F}_t] = \tilde{\mathbb{E}}[\mathbf{D}_t^j(B_T^i)|\mathcal{F}_t] = \delta_{ij}.$$

It follows that

$$\sum_{1 \leq k \leq J} \mathbf{D}_t^j(B_T^k) \frac{\partial}{\partial x^k} G(B_T) = \frac{\partial}{\partial x^j} G(B_T).$$

Now we can put everything together to obtain

$$\begin{aligned} \tilde{\sigma}^{ij} &= \tilde{\mathbb{E}}[\mathbf{D}_t^j(f^i)|\mathcal{F}_t] + \gamma \widetilde{\text{Cov}} \left(f^i, \sum_{1 \leq k \leq J} \mathbf{D}_t^j(f^k) \frac{\partial}{\partial x^k} G(B_T) \middle| \mathcal{F}_t \right) \\ &= \delta_{ij} + \gamma \widetilde{\text{Cov}} \left(B_T^i, \frac{\partial}{\partial x^j} G(B_T) \middle| \mathcal{F}_t \right). \end{aligned}$$

□

As an important corollary we state the following result showing that in the framework of the Bachelier model a large class of *convex* (in appropriate sense) contingent claims $G(f)$ is replicable. For instance this includes a convex combination of *long* positions in European calls written on each individual asset.

Corollary 4.1. *Assume conditions of Lemma 4.4 . Let*

$$G = G(x^1, \dots, x^J) = \sum_{j=1}^J c_j \varphi_j(x^j),$$

where $c_j \geq 0$, $c_j \in \mathcal{R}$ and $\varphi_j = \varphi_j(x)$ are one-dimensional convex functions. Then for any contingent claim g of the form

$$(4.18) \quad g = G(f) = G(B_T),$$

hedging strategy H exists and is defined uniquely $d\mathbb{P}[\omega] \times dt$ -a.s.

Proof. Let us observe that by the assumption the function G is convex in each x^j direction, that is its partial derivatives $\frac{\partial G}{\partial x^j} = c_j \frac{d\varphi_j(x^j)}{dx}$ are non-decreasing

functions in each x^j direction. Hence for any $\mathbf{x}_1 = (x_1^1, \dots, x_1^J) \in \mathcal{R}^J$, $\mathbf{x}_2 = (x_2^1, \dots, x_2^J) \in \mathcal{R}^J$, and for any $1 \leq i, j \leq J$

$$(x_1^i - x_2^i) \left(\frac{\partial G(\mathbf{x}_1)}{\partial x^j} - \frac{\partial G(\mathbf{x}_2)}{\partial x^j} \right) = (x_1^i - x_2^i) c_j \left(\frac{d\varphi^j(x_1^j)}{dx} - \frac{d\varphi^j(x_2^j)}{dx} \right) \geq 0,$$

and therefore

$$(4.19) \quad c_j (B_T^i(\omega_1) - B_T^i(\omega_2)) \left(\frac{d\varphi^j(B_T(\omega_1))}{dx} - \frac{d\varphi^j(B_T(\omega_2))}{dx} \right) \geq 0, \\ d\tilde{\mathbb{P}}[\omega_1] \otimes d\tilde{\mathbb{P}}[\omega_2] - a.s.$$

Condition (4.19) means that the random variables B_T^i and $c_j \frac{d\varphi^j(B_T)}{dx}$ are *co-monotone*. It follows that

$$(4.20) \quad c_j \int_{\Omega_1 \times \Omega_2} (B_T^i(\omega_1) - B_T^i(\omega_2)) \left(\frac{d\varphi^j(B_T(\omega_1))}{dx} - \frac{d\varphi^j(B_T(\omega_2))}{dx} \right) d\tilde{\mathbb{P}}[\omega_1] \times d\tilde{\mathbb{P}}[\omega_2] \geq 0.$$

By Fubini's theorem (4.20) is equal to

$$\begin{aligned} & c_j \int_{\Omega_1} \left(B_T^i(\omega_1) - \int_{\Omega_2} B_T^i(\omega_2) d\tilde{\mathbb{P}}[\omega_2] \right) \\ & \quad \times \left(\frac{d\varphi^j(B_T(\omega_1))}{dx} - \int_{\Omega_2} \frac{d\varphi^j(B_T(\omega_2))}{dx} d\tilde{\mathbb{P}}[\omega_2] \right) d\tilde{\mathbb{P}}[\omega_1] \\ &= \tilde{\mathbb{E}} \left[(B_T^i - \tilde{\mathbb{E}}[B_T^i]) \left(c_j \frac{d\varphi^j(B_T)}{dx} - \tilde{\mathbb{E}} \left[c_j \frac{d\varphi^j(B_T)}{dx} \right] \right) \right] \\ &= \tilde{\mathbb{E}} \left[(B_T^i - \tilde{\mathbb{E}}[B_T^i]) \left(\frac{\partial G(B_T)}{\partial x^j} - \tilde{\mathbb{E}} \left[\frac{\partial G(B_T)}{\partial x^j} \right] \right) \right] = \widetilde{\text{Cov}}(B_T^i, \frac{\partial G(B_T)}{\partial x^j}). \end{aligned}$$

Hence the unconditional covariance of B_T^i and $\frac{\partial G(B_T)}{\partial x^j}$ is non-negative. By the similar argument one can show that the conditional covariance of B_T^i and $\frac{\partial G(B_T)}{\partial x^j}$ is

$$\widetilde{\text{Cov}} \left(B_T^i, \frac{\partial G(B_T)}{\partial x^j} \middle| \mathcal{F}_t \right) \geq 0.$$

We now notice that because of the independence of individual Brownian Motions $(B^i)_{1 \leq i \leq J}$ of the J -dimensional Brownian Motion B

$$\tilde{\sigma}_t^{ij} = \delta_{ij} + \gamma \widetilde{\text{Cov}} \left(B_T^i, \frac{\partial G(B_T)}{\partial x^j} \middle| \mathcal{F}_t \right) = 0 \quad d\mathbb{P}[\omega] \times dt\text{-a.s., when } i \neq j,$$

and

$$\tilde{\sigma}_t^{ij} = \delta_{ij} + \gamma \widetilde{\text{Cov}} \left(B_T^i, \frac{\partial G(B_T)}{\partial x^j} \middle| \mathcal{F}_t \right) \geq 1 \quad d\mathbb{P}[\omega] \times dt\text{-a.s.}, \text{ when } i = j.$$

Therefore the covariance matrix $\tilde{\sigma}$ is a diagonal matrix with non-zero entries on the diagonal and hence invertible $d\mathbb{P}[\omega] \times dt\text{-a.s.}$

Finally we deduce from Theorem 3.2 and Proposition 4.1 that a hedging strategy H exists and is defined uniquely $d\mathbb{P}[\omega] \times dt\text{-a.s.}$ \square

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